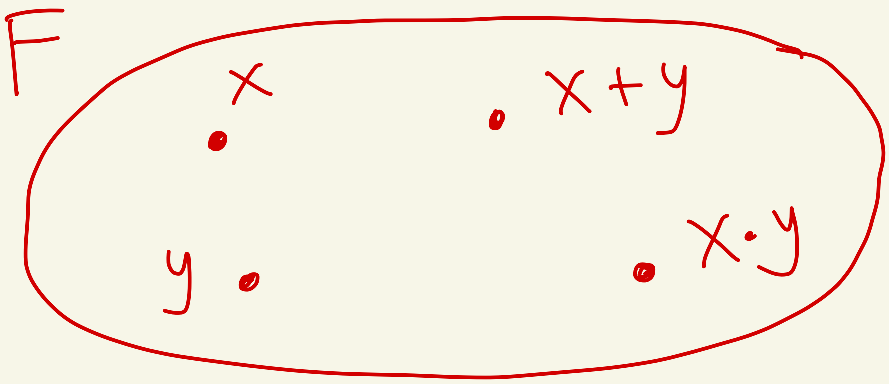
Topic 6 - Vector Spaces

HW 6 Topic - Vector Spaces

We are going to generalize what a scalar/number is. Field

Then we will generalize what a vector is. vector space

Def: A field F is a set of objects with two operations $+$ and $\cdot$ such that for any two elements x and y in the field F we have that there exist unique elements $x+y$ and $x \cdot y$ in the field F.



people say that F is "closed" under $+$ and $\cdot$

Also the following three properties must hold.

Also the following three properties must hold. If a and b are in F, then

(F1) If

$$a+b = b+a$$
$$a \cdot b = b \cdot a$$

commutativity

$$(a+b)+c = a+(b+c)$$
$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

associativity

$$a \cdot (b+c) = a \cdot b + a \cdot c$$
$$(b+c) \cdot a = b \cdot a + c \cdot a$$

distributive properties

(F2) There exist distinct elements 0 and 1 in F where

$$x + 0 = 0 + x = x$$

$$x \cdot 1 = 1 \cdot x = x$$

for all x in F.

0 is called an additive identity
1 is called a multiplicative identity

(F3) For each x in F, there exists a unique element in F, written as $-x$, where

$$x + (-x) = (-x) + x = 0$$

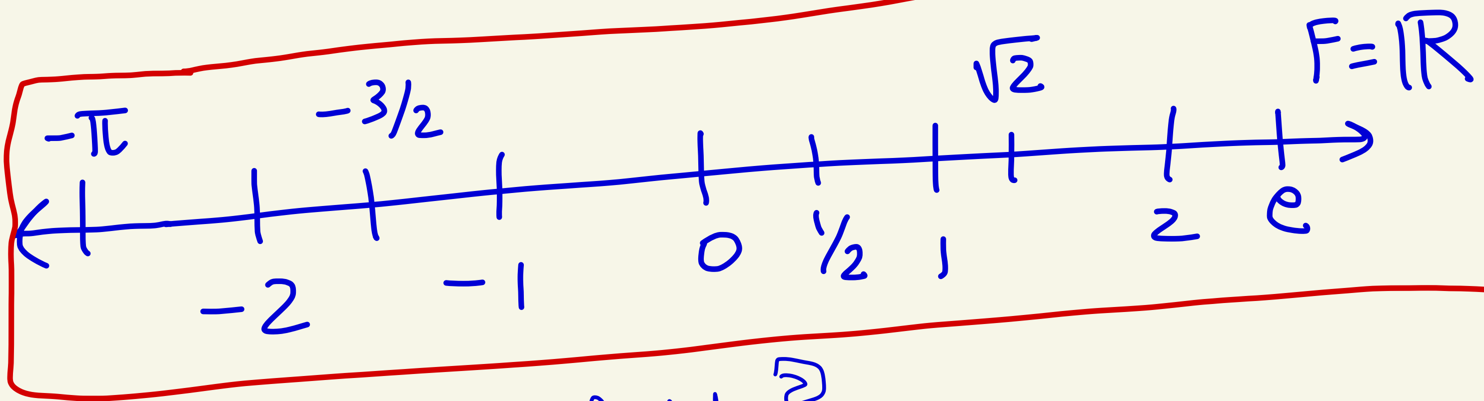
$-x$ is called the additive inverse of x

If $x \neq 0$, there exists a unique element in F, written as x^{-1}, where

$$x \cdot x^{-1} = x^{-1} \cdot x = 1$$

x^{-1} is called the multiplicative inverse of x

Ex: $F = \mathbb{R}$, the set of real numbers, is a field using the usual $+$ and $\cdot$.



Why is $\mathbb{R}$ a field?

- Adding and multiplying real numbers gives a real number.

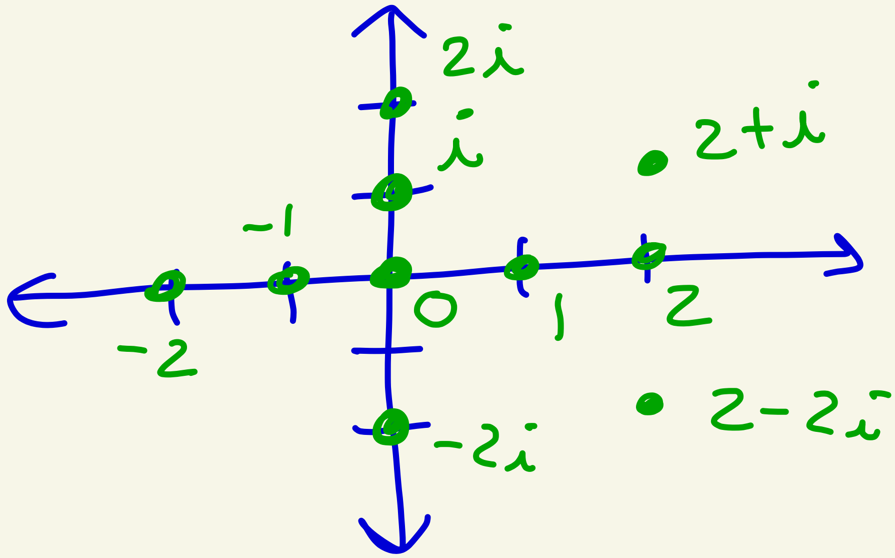
① All the properties from F1 are true in $\mathbb{R}$.

② $\mathbb{R}$ has elements 0 and 1 that behave as in F2.

③ We have F3 is true.

Note: In our class, $\mathbb{R}$ is the only field that we will use. But let's see some others just to see.

<u>Ex:</u> The set of complex numbers $\mathbb{C}$ is a field.



We won't use this field in this class.

<u>Ex:</u> There even exist fields that are finite in size. You get these by "modular arithmetic".

For our class, we will always use $\mathbb{R}$ as our field.

But I will state theorems for general fields.

Now we generalize what a "vector" is.

$\boxed{\text{Make pg 7 - 8 a handout.}}$

Def: A <u>vector space</u> V over a field F consists of a set of "vectors" V with two operations, "vector addition" $+$ and "scalar multiplication" $\cdot$, such that the following hold:

① If $\vec{v}$ and $\vec{w}$ are in V, then $\vec{v}+\vec{w}$ is in V. — V is closed under $+$

② If $\vec{v}$ is in V and α is in F, then $\alpha\cdot\vec{v}$ is in V. — V is closed under scaling $\cdot$

③ If $\vec{v}$ and $\vec{w}$ are in V, then $\vec{v}+\vec{w}=\vec{w}+\vec{v}$ — commutative property

④ If $\vec{v},\vec{w},\vec{z}$ are in V, then $\vec{v}+(\vec{w}+\vec{z})=(\vec{v}+\vec{w})+\vec{z}$ — associative property

⑤ There exists a unique vector $\vec{0}$ in V where $\vec{v}+\vec{0}=\vec{0}+\vec{v}=\vec{v}$ for all $\vec{v}\in V$. — additive identity $\vec{0}$

(6) For every vector $\vec{v}$ in V there exists a unique vector denoted by $-\vec{v}$ where $\vec{v}+(-\vec{v}) = (-\vec{v})+\vec{v} = \vec{0}$

additive inverses

(7) If $\vec{v}$ is in V and 1 is the multiplicative identity of F, then $1 \cdot \vec{v} = \vec{v}$.

scaling by 1 doesn't change the vector

(8) If $\vec{v}$ is in V and α, β are in F, then $(\alpha\beta)\cdot\vec{v} = \alpha\cdot(\beta\cdot\vec{v})$

(9) If $\vec{v}$ and $\vec{w}$ are in V and α is in F, then $\alpha\cdot(\vec{v}+\vec{w}) = \alpha\cdot\vec{v}+\alpha\cdot\vec{w}$

(10) If $\vec{v}$ is in V and α, β are in F, then $(\alpha+\beta)\cdot\vec{v} = \alpha\cdot\vec{v}+\beta\cdot\vec{v}$

distrib-utive prop's

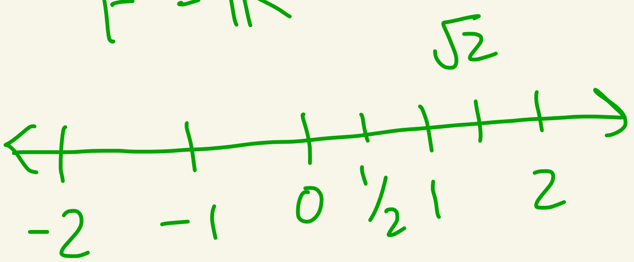
$\boxed{\text{Ex:}}$ Let $V = \mathbb{R}^n$ and $F = \mathbb{R}$

$\mathbb{R}^n$ is a vector space over the field $\mathbb{R}$ using the usual vector addition and scalar multiplication.

$\boxed{\text{Ex:}}$ $\boxed{n=2}$

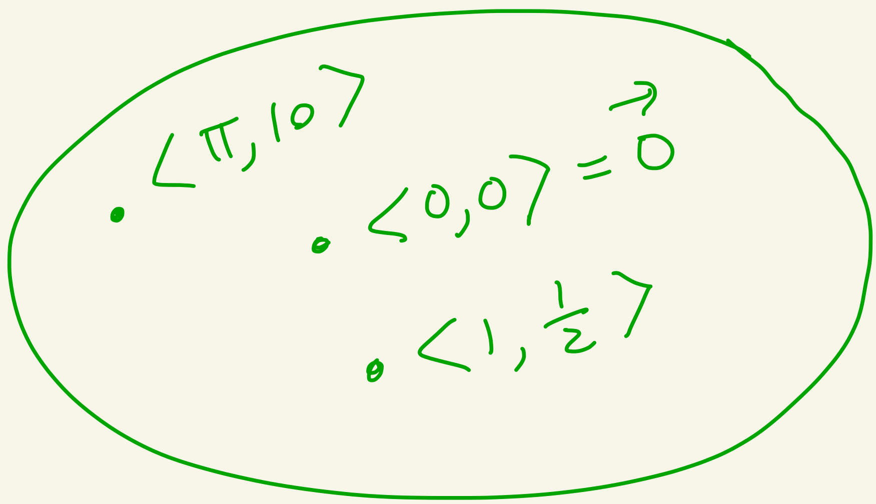
<u>Field</u>

$F = \mathbb{R}$



<u>Vectors</u>

$V = \mathbb{R}^2$



<u>vector addition:</u>

$$\langle 1, \tfrac{1}{2} \rangle + \langle 0, -5 \rangle = \langle 1, -\tfrac{9}{2} \rangle$$

<u>scalar multiplication:</u>

$$5 \cdot \langle 1, -2 \rangle = \langle 5, -10 \rangle$$

One can check that this example satisfies all 10 properties of being a vector space. Some we did in class and HW in earlier topics.

Ex: Let

$$V = M_{2,2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \text{ are real numbers} \right\}$$

$$= \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 5 & \pi \end{pmatrix}, \begin{pmatrix} \sqrt{2} & \frac{1}{2} \\ 5 & 3 \end{pmatrix}, \ldots \right\}$$

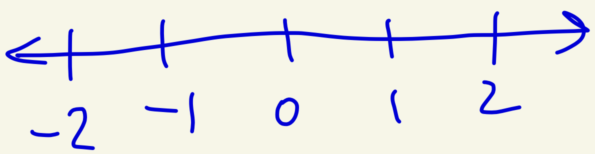
these are the "vectors"

infinitely many more

and $F = \mathbb{R}$

scalars

field $F = \mathbb{R}$



Vectors $V = M_{2,2}$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ • $\begin{pmatrix} 5 & -1 \\ \frac{1}{2} & 0 \end{pmatrix}$ • $\begin{pmatrix} \pi & \pi \\ \pi & \pi \end{pmatrix}$ •

We will use the usual addition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

and scalar multiplication

$$\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

One can check that the 10 vector space properties hold.

Here the zero vector is

$$\vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and the additive inverse of $\vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $-\vec{v} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$

So, $V = M_{2,2}$ is a vector space over the field $F = \mathbb{R}$.

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Ex: Pick some integer $n \geq 0$

(So, n can be $0, 1, 2, 3, 4, \ldots$)

Let V be the set of all polynomials of degree $\leq n$, denoted by P_n.

So,

$$V = P_n = \left\{ \overbrace{a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n}^{\boxed{\text{"vectors"}}} \;\middle|\; \begin{array}{c} a_0, a_1, \ldots, a_n \\ \text{are real} \\ \text{numbers} \end{array} \right\}$$

Let $F = \mathbb{R}$. $\leftarrow \boxed{\text{scalars}}$

Define vector addition as the usual polynomial addition

That is,

$$(a_0+a_1x+\dots+a_nx^n)+(b_0+b_1x+\dots+b_nx^n)$$
$$=(a_0+b_0)+(a_1+b_1)x+\dots+(a_n+b_n)x^n$$

Scalar multiplication is

$$\alpha(a_0+a_1x+\dots+a_nx^n)$$
$$=(\alpha a_0)+(\alpha a_1)x+\dots+(\alpha a_n)x^n$$

Two polynomials are defined to be equal if they have the same coefficients. That is,

$$a_0+a_1x+\dots+a_nx^n=b_0+b_1x+\dots+b_nx^n$$

if and only if

$$a_0=b_0,\ a_1=b_1,\ \dots,\ a_n=b_n$$

Here,

$$\vec{0} = 0 + 0x + 0x^2 + \dots + 0x^n$$

and

$$-(a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n)$$
$$= (-a_0) + (-a_1)x + (-a_2)x^2 + \dots + (-a_n)x^n$$

One can verify that properties 1 – 10 are true and hence $V = P_n$ is a vector space over $F = \mathbb{R}$.

Ex: $F = \mathbb{R}$ ← scalars

vectors

$V = P_3$

$$= \left\{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \;\middle|\; \begin{array}{l} a_0, a_1, a_2, a_3 \\ \text{are in } \mathbb{R} \end{array} \right\}$$

$$= \{ 1 - x, \ \vec{0}, \ 10, \ 1 + x - x^2 + x^3, \ x^3, \ \ldots \}$$

$\boxed{1 + (-1)x + 0x^2 + 0x^3}$

$\boxed{0 + 0x + 0x^2 + 0x^3}$

$\boxed{10 + 0x + 0x^2 + 0x^3}$

$\boxed{0 + 0x + 0x^2 + 1 \cdot x^3}$

infinitely many more

Examples of adding & scaling:

$$(1 - x) + (1 + x - x^2 + x^3) = 2 - x^2 + x^3$$

$$5(1 + x - x^2 + x^3) = 5 + 5x - 5x^2 + 5x^3$$

Notice that P_3 behaves like $\mathbb{R}^4$. The powers of x are like placeholders.

$$(1+2x-x^2+x^3)+(5-x+x^2+2x^3)$$

$$= 6+x \qquad +3x^3$$

This is like

$$\langle 1,2,-1,1\rangle + \langle 5,-1,1,2\rangle$$

$$= \langle 6,1,0,3\rangle$$

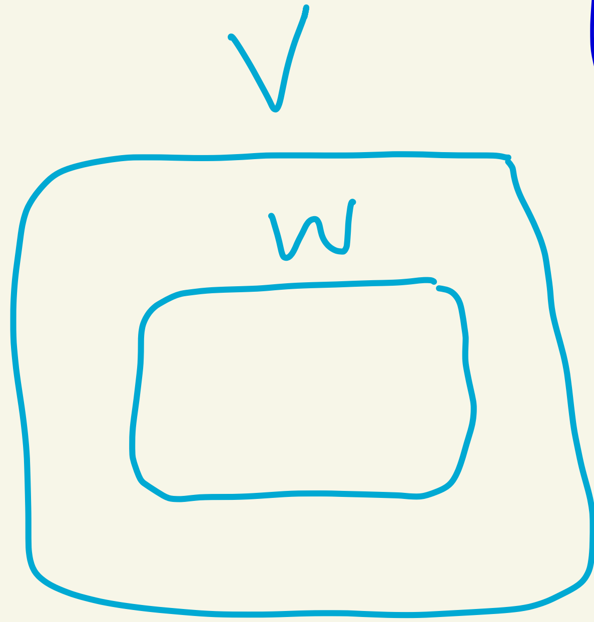
and scaling

$$3\cdot(1+x-x^2+5x^3) = 3+3x-3x^2+15x^3$$

thats like

$$3\langle 1,1,-1,5\rangle = \langle 3,3,-3,15\rangle$$

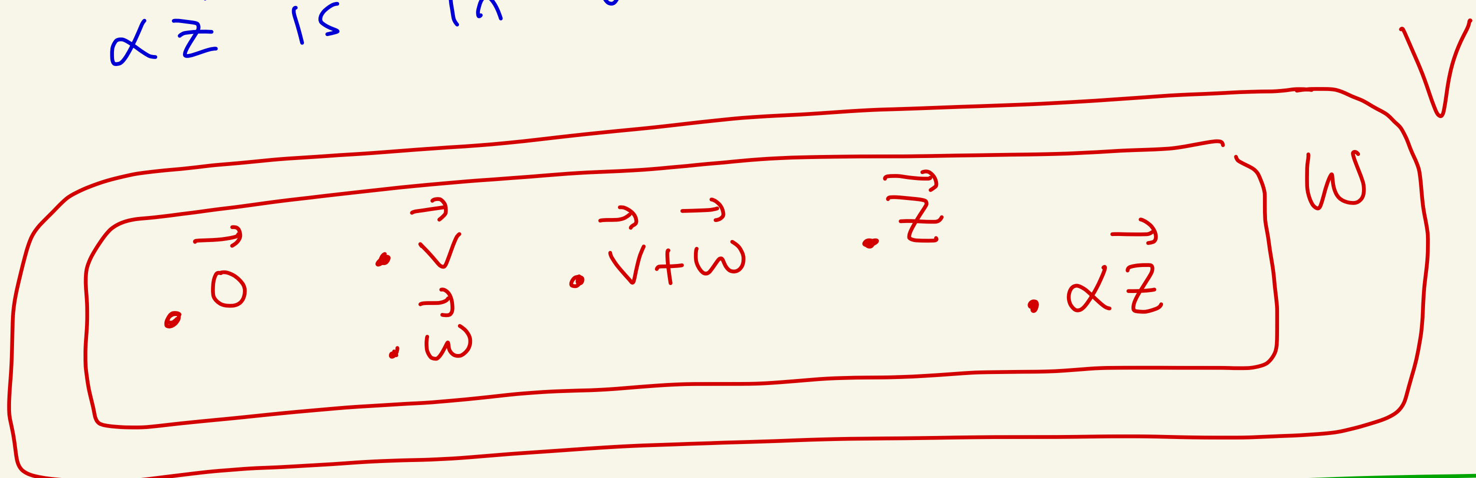
Def: Let V be a vector space over a field F. Let W be a subset of V. We say that W is a <u>subspace</u> of V if the following three conditions hold:



① $\vec{0}$ is in W.

② If $\vec{v}$ and $\vec{w}$ are in W, then $\vec{v}+\vec{w}$ is in W. — W is closed under vector addition

③ If $\vec{z}$ is in W and α is in F, then $\alpha\vec{z}$ is in W. — W is closed under scaler multiplication



<u>Note:</u> One can show that if W is a subspace of V if and only if W itself is a vector space living inside of V.

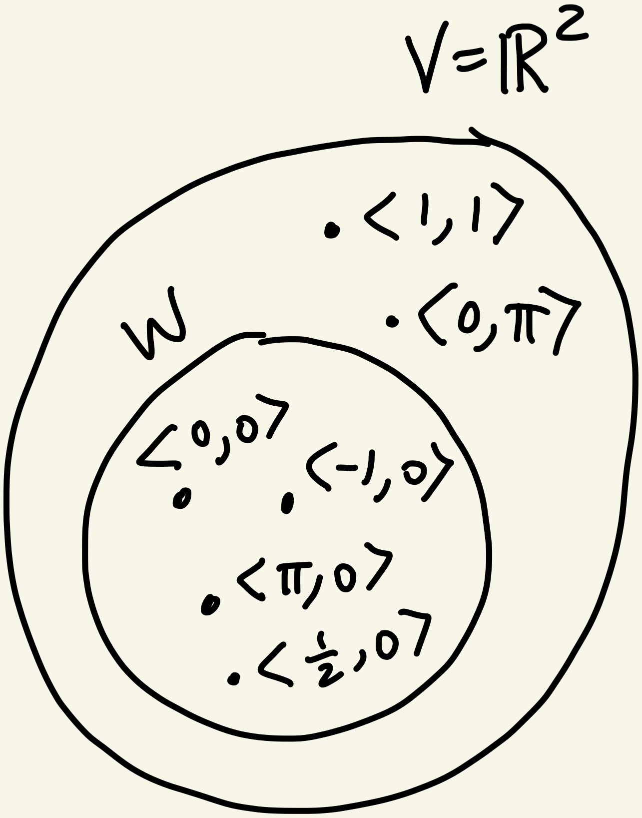
Ex: Consider the vector space $V = \mathbb{R}^2$ over the field $F = \mathbb{R}$.

Let

$$W = \{ \langle x, 0 \rangle \mid x \in \mathbb{R} \}$$

$$= \{ \underbrace{\langle 0, 0 \rangle}_{x=0}, \underbrace{\langle -1, 0 \rangle}_{x=-1}, \underbrace{\langle \pi, 0 \rangle}_{x=\pi}, \ldots \}$$

infinitely many more



Let's prove that W is a subspace of V.

proof:

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(1) Set $x=0$ in $\langle x,0\rangle$ and we get that $\langle 0,0\rangle = \vec{0}$ is in W.

(2) Let $\vec{v}, \vec{w}$ be in W.
Then, $\vec{v} = \langle x_1, 0\rangle$ and $\vec{w} = \langle x_2, 0\rangle$
where $x_1, x_2 \in \mathbb{R}$.
Then, $\vec{v} + \vec{w} = \langle x_1 + x_2, 0\rangle$
which is an element of W.

(3) Let $\vec{z}$ be in W and α be in $F = \mathbb{R}$.
Since $\vec{z}$ is in W we know that
$\vec{z} = \langle x, 0\rangle$ where $x \in \mathbb{R}$.
Then, $\alpha\vec{z} = \alpha\langle x, 0\rangle = \langle \alpha x, 0\rangle$
which is an element of W.

By (1), (2), and (3) we have that W is a subspace of $V = \mathbb{R}^2$ ∎

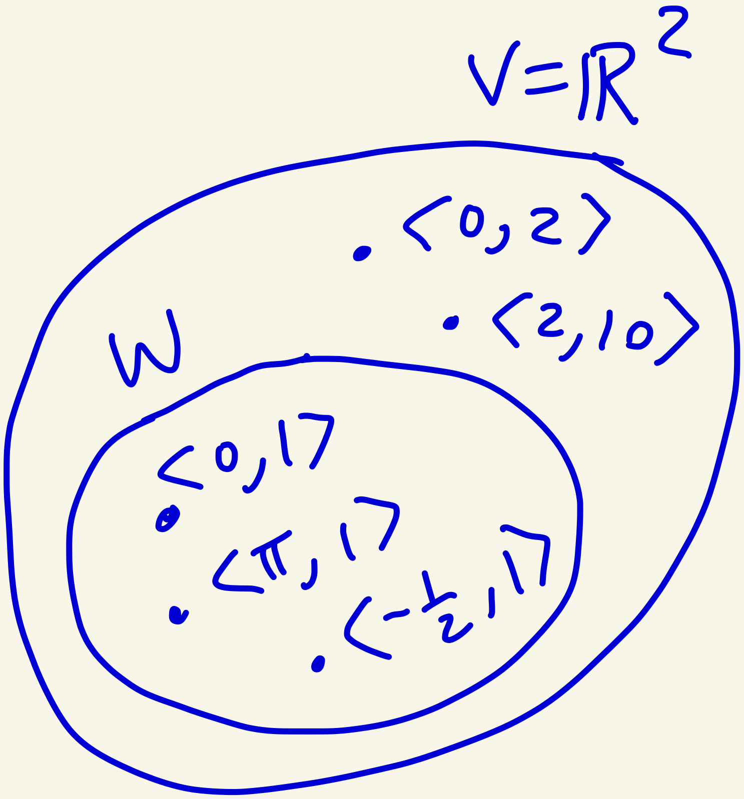
Ex: Consider the vector space $V = \mathbb{R}^2$ over $F = \mathbb{R}$.

Consider

$$W = \{ \langle x, 1 \rangle \mid x \in \mathbb{R} \}$$

$$= \{ \underbrace{\langle 0, 1 \rangle}_{x=0}, \underbrace{\langle \pi, 1 \rangle}_{x=\pi}, \underbrace{\langle -\tfrac{1}{2}, 1 \rangle}_{x=-\frac{1}{2}}, \ldots \}$$

infinitely many more



It turns out that W is <u>not</u> a subspace of $V=\mathbb{R}^2$.

<u>For example:</u>

(1) Note that $\vec{0}=\langle 0,0\rangle$ is not of the form $\langle x,1\rangle$. Thus, $\vec{0}\notin W$. So W is not a subspace of $V=\mathbb{R}^2$.

One could also show that (2) or (3) don't hold for W.

<u>For example:</u>

(2) Let $\vec{v}=\langle 2,1\rangle$ and $\vec{w}=\langle 3,1\rangle$. Then $\vec{v},\vec{w}$ are both in W.
However,
$$\vec{v}+\vec{w}=\langle 2,1\rangle+\langle 3,1\rangle = \langle 5,2\rangle$$
which isn't in W.

Thus, condition (2) doesn't hold and W is not a subspace of $V=\mathbb{R}^2$.

Ex: Let $F = \mathbb{R}$ and

$$V = M_{2,2} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} 1 & 2 \\ 5 & \pi \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ldots \right\}$$

$\vec{0}$ in $M_{2,2}$

infinitely many more

We talked about how $M_{2,2}$ is vector space where vector addition is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$$

and scalar multiplication is given by

$$\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

α in $F = \mathbb{R}$

Let

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; \begin{array}{l} d = a+b \\ a,b,c,d \in \mathbb{R} \end{array} \right\}$$

$$= \left\{ \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}}_{2=1+1}, \underbrace{\begin{pmatrix} 5 & -10 \\ \frac{1}{2} & -5 \end{pmatrix}}_{-5=5-10}, \underbrace{\ldots}_{\text{infinitely many more}} \right\}$$

Before we prove W is a subspace:

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$ because $0 = 0+0$.

$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 5 & -10 \\ \frac{1}{2} & -5 \end{pmatrix} \in W$ and $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 5 & -10 \\ \frac{1}{2} & -5 \end{pmatrix}$

$$= \begin{pmatrix} 6 & -9 \\ \frac{3}{2} & -3 \end{pmatrix} \in W$$

because $-3 = 6-9$

$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \in W$ and $3 \cdot \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix} \in W$

because $6 = 3+3$

Let's prove that W is a subspace of $V = M_{2,2}$.

<u>proof:</u> We need to check the 3 criteria from the previous theorem.

① Is $\vec{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ in W?

Yes, if we set $a = b = c = d = 0$ then $\vec{0} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\underbrace{d = a+b}_{0 = 0+0}$

② Is W closed under vector addition?

Let $\vec{v}$ and $\vec{w}$ be in W.

Then, $\vec{v} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

where $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in \mathbb{R}$

and $\underbrace{d_1 = a_1 + b_1}_{\text{since } \vec{v} \in W}$ and $\underbrace{d_2 = a_2 + b_2}_{\text{since } \vec{w} \in W}$

Then,

$$\vec{v}+\vec{w} = \begin{pmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{pmatrix}$$

Adding $d_1 = a_1 + b_1$ and $d_2 = a_2 + b_2$
gives $d_1 + d_2 = a_1 + b_1 + a_2 + b_2$

Regrouping gives

$$\boxed{d_1 + d_2 = (a_1 + a_2) + (b_1 + b_2)} \quad (*)$$

$(*)$ tells us that $\vec{v}+\vec{w}$ is in W.

So, W is closed under vector addition.

③

(3) Let's show that W is closed under scalar multiplication.

Let $\vec{z} \in W$ and $\underbrace{\alpha \in \mathbb{R}}_{F = \mathbb{R}}$

Since $\vec{z} \in W$ we know that

$\vec{z} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{R}$ and $d = a + b$.

Then,

$$\alpha \vec{z} = \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

Multiplying $d = a + b$ by α gives

$$(\alpha d) = (\alpha a) + (\alpha b) \quad (**)$$

And $(**)$ tells us that

$\alpha \vec{z} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$ is in W.

Thus, W is closed under scalar multiplication.

Since W satisfies properties 1, 2, and 3 above, W is a subspace of $V = M_{2,2}$. $\blacksquare$